Discrete Mathematics Quiz 1

2022 - 2023 春夏学期

Shd0wash

2023 年 6 月 2 日

1. Determine whether the following statements are true or false. (30%)
(**5 points** for a correct answer, **0 points** for a blank answer, **-2 points for an incorrect answer**)

(a) The following two propositions are logicially equivalent:

$$\forall x P(x) \vee \exists x Q(x), \forall x \exists y (P(x) \vee Q(y))$$

where all quantifiers have the same nonempty domain.
(b) If A, B *and* C are sets, then $(A - C) - (B - C) = A - B$.
(c) The set of real numbers that are solutions of cubic equations $ax^3 + bx^2 + cx + d = 0$, where a, b, c, d are integers, is uncountable.
(d) If $P(A) \in P(B)$, then $A \in B$.
(e) The formula $\neg(p \to q) \wedge q$ is a tautology.
(f) There exists a one-to-one function from $\mathbb{R}$ to $\mathbb{Z} \times \mathbb{Z}$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ is the set of integers.

2. Single choice questions. (5%)

Which of the following statements is true?
A. $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$
B. $\exists x(P(x) \wedge Q(x)) \equiv \exists x P(x) \wedge \exists x Q(x)$
C. $\forall x(P(x) \to Q(x)) \equiv \forall x P(x) \to \forall x Q(x)$
D. $\exists x(P(x) \to Q(x)) \equiv \exists x P(x) \to \exists x Q(x)$

3. Write a proposition equivalent to $(p \wedge q)$,
(a) using only $p, q, \neg$, and the connective $\vee$. (5%)
(b) using only p, q, and the connective $\downarrow$. (5%)
(“ $\downarrow$ ” represents NOR. The proposition p NOR q is true when both p and q are false, and it is false otherwise)

4. Put the functions below in order so that each function is big-O of the next function on the list. (8%)

$f_1(n) = (1.2)^n$	$f_2(n) = 7n^6 + n + 323$	$f_3(n) = (\log n)^3$	$f_4(n) = 3^n$	$f_5(n) = \log(\log n)$
$f_6(n) = n^2(\log n)^3$	$f_7(n) = 3^n(n^3 + 1)$	$f_8(n) = n^3 + n(\log n)^2$	$f_9(n) = 1000000$	$f_{10}(n) = 10n!$

5.

(a) Show the full conjunctive normal form of $(p \leftrightarrow \neg r) \rightarrow (q \leftrightarrow r)$. (6%)

(b) Show the full disjunctive normal form of $(p \leftrightarrow \neg r) \rightarrow (q \leftrightarrow r)$. (6%)

6. Let S be a subset of rational numbers, satisfying the following conditions:

(1) If $a \in S, b \in S, then\ \ a + b \in S, ab \in S$;

(2) For any rational number r, there is one and only one of the three relationship holds:

$$r \in S, \ \ -r \in S, \ \ r = 0.$$

Prove that $S = \mathbb{Q}^+$ ($\mathbb{Q}^+$ is the set of positive rational numbers).

7. Use induction to prove that for all nonnegative integer n,

(a) $f_{5n} \equiv 0 \pmod 5$, (6%)

(b) $f_n^2 + f_{n+1}^2 = f_{2n+1}$, (9%)

where f_i denotes the ith Fibonacci number.

8. There are 2 piles of stones that begins with 115 stones and 125 stones respectively. Suppose that two people play a game taking turns removing one, two, or three stones at a time from one of the piles. The person who removes the last stone wins the game. Show that the first player can win the game no matter what the second player does. (8%)